Announcements

1) HW 4 redone,
   now due 3/27

2) Strang's Linear
   Algebra - online!

Example 1: Back to $x+2$

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = x+2$$

$$\tilde{f} : \mathbb{R} \to \mathbb{R}$$

$$\tilde{f}(x) = fl(x) \oplus 2$$

Show $\tilde{f}$ is stable

Unlike backwards stability, checking whether an algorithm is stable requires an "inspired" choice of $\tilde{x}$ since we only need

$$\frac{\|f(\tilde{x}) - \tilde{f}(x)\|}{\|f(x)\|} = O(\varepsilon_{\text{machine}})$$

instead of $f(\tilde{x}) = \tilde{f}(x)$.

Guess: $\tilde{x} = x$.

Then

$$\frac{|x - \tilde{x}|}{|x|} = 0$$

which is trivially

$O(\varepsilon_{machine})$!

We need to check

$$\frac{|\tilde{f}(x) - f(\tilde{x})|}{|f(x)|}$$

$$= \frac{|\tilde{f}(x) - f(x)|}{|f(x)|}$$

$$= O(\varepsilon_{machine})$$

Calculate and check.

Recall: $\exists\ \varepsilon_1, \varepsilon_2$ with

$|\varepsilon_1|, |\varepsilon_2| < \varepsilon_{machine}$

and

$$fl(x) = x(1+\varepsilon_1),$$

$$fl(x) \oplus 2 = (fl(x)+2)(1+\varepsilon_2)$$

$$\tilde{f}(x) - f(x)$$

$$= (fl(x) \oplus 2) - (x+2)$$

$$= (1+\varepsilon_2)(fl(x)+2) - (x+2)$$

$$= (1+\varepsilon_2)((1+\varepsilon_1)x+2) - (x+2)$$

$$= (1+\varepsilon_2)(1+\varepsilon_1)x + \cancel{2} + 2\varepsilon_2$$

$$- x - \cancel{2}$$

$$= \cancel{x} + \varepsilon_1 x + \varepsilon_2 x + \varepsilon_1\varepsilon_2 x + 2\varepsilon_2 - \cancel{x}$$

$$= (\varepsilon_1 + \varepsilon_2 + \varepsilon_1 \varepsilon_2) x + 2 \varepsilon_2$$

Then since

$$|\varepsilon_1 + \varepsilon_2 + \varepsilon_1 \varepsilon_2| = O(\varepsilon_{machine})$$

$$(\varepsilon_{machine} \to 0)$$

and

$$|\varepsilon_2| = O(\varepsilon_{machine}),$$

we get

$$|\tilde{f}(x) - f(x)|$$

$$= |(\varepsilon_1 + \varepsilon_2 + \varepsilon_1 \varepsilon_2)x + 2\varepsilon_2|$$

$$= |x+2|\, O(\varepsilon_{machine}).$$

(see HW 4:

$$\frac{1}{1+O(\varepsilon_{machine})} = 1 + O(\varepsilon_{machine})$$

Then

$$\frac{|\tilde{f}(x)-f(x)|}{|f(x)|}$$

$$= \frac{\cancel{|x+2|}\, O(\varepsilon_{machine})}{\cancel{|x+2|}}$$

$$= O(\varepsilon_{machine})$$ ✓

Example 2: (more stability)

Given $v, w \in \mathbb{C}^n$,

Computing

1) $v^* w$ is backwards stable

2) $v w^*$ is stable, but not backwards stable

Example 3: (unstable)

Computing the eigenvalues of a matrix via:

1) Finding the characteristic polynomial

2) Taking the coefficients as data

3) Finding the zeros of the characteristic polynomial.

This procedure is ill-conditioned (denominator with $p'(x_j)$) - no stable algorithm can exist for an ill-conditioned problem.

Notation: (little o)

We say $f(x) = o(g(x))$

if either

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 0$$

our use

– or –

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0 .$$

Example 4: (little o)

a) $p(x) =$ any polynomial,

$$p(x) = o(e^x) \text{ as } x \to \infty$$

(l'Hopital's rule)

b) $f(x) = x$, $g(x) = 2x$

$f(x) = O(g(x))$ but

not $o(g(x))$ since $\frac{f(x)}{g(x)} = \frac{1}{2}$

$(x \neq 0)$

Theorem: (Backwards stability and conditioning)

Let $f : X \to Y$ be a problem and let $\tilde{f}$ be a backwards stable algorithm $\tilde{f} : X \to Y$ for f and some $\varepsilon_{machine}$.

Then for all $x \in X$,

$$\frac{\| f(x) - \tilde{f}(x) \|}{\| f(x) \|} = O(\kappa(x)\, \varepsilon_{machine})$$

proof: Will now write

$x - \tilde{x}$ for δx and

$f(x) - f(\tilde{x})$ for δf.

By backwards stability,

there is some $\tilde{x} \in X$,

$$\frac{\|x-\tilde{x}\|}{\|x\|} = O(\varepsilon_{machine}),$$

$$\tilde{f}(x) = f(\tilde{x}).$$

By definition of $\kappa(x)$,

$$\frac{\|f(x)-f(\tilde{x})\|}{\|f(x)\|} \cdot \frac{\|x\|}{\|x-\tilde{x}\|}$$

$\leq \kappa(x)$, so

$$\frac{\|f(x)-f(\tilde{x})\|}{\|f(x)\|} \leq \frac{\|x-\tilde{x}\|}{\|x\|} \kappa(x)$$

$$= \kappa(x) O(\varepsilon_{machine})$$

Substituting $\tilde{f}(x)$ for $f(\tilde{x})$,

$$\frac{\|f(x) - \tilde{f}(x)\|}{\|f(x)\|} \leq K(x)\, O(\varepsilon_{machine})$$

$$= O(K(x)\,\varepsilon_{machine})$$

$\square$